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Fredholm's minors of arbitrary order: their representations as a determinant of resolvents and in terms of free fermions and an explicit formula for their functional derivative

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Abstract

We study the Fredholm minors associated with a Fredholm equation of the second type. We present a couple of new linear recursion relations involving the nth and $(n-1)$th minors, whose solution is a representation of the nth minor as an $n \times n$ determinant of resolvents. The latter is given a simple interpretation in terms of a path integral over non-interacting fermions. We also provide an explicit formula for the functional derivative of a Fredholm minor of order n with respect to the kernel. Our formula is a linear combination of the nth and the $(n \pm 1)$th minors.

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1. Introduction

The ubiquity of linear integral equations, and in particular of Fredholm equations (FE) [1–3], in mathematical physics, and more broadly in analysis, cannot be overstated. Thus, new results in this classical field, such as those presented in this paper (equations (2.11) and (4.9)), should be of some interest: equation (2.11) expresses Fredholm's nth minor (1.8) as a determinant of resolvents (A.2) of the equation, and equation (4.9) is a formula which gives the functional derivative of the nth minor (with respect to the kernel) in terms of the nth and $(n \pm 1)$th minors in closed form. In addition, in section 3 we make a quantum-field-theoretic interpretation of our determinantal representation (2.11) as a correlator of non-interacting fermions.

To get oriented, let us briefly recall the basic definitions and facts of the Fredholm theory, relevant to our discussion. (We have adopted throughout this paper the conventions and notation of chapter 2 of [3].) Thus, consider a Fredholm integral equation of the second type in the unknown function $\phi(x)$,

$$\phi(x) = f(x) + \lambda \int_\Omega N(x, y)\phi(y)\,\mathrm{d}y \tag{1.1}$$

with kernel $N(x, y)$ and given function $f(x)$. For simplicity, we shall take $N(x, y)$ and $f(x)$ as real functions. The generalization to the complex case is straightforward. The complex variable λ is the spectral parameter of the equation, and Ω is the domain on which the equation is defined. To be concrete, we shall take Ω as a compact domain of the N-dimensional Euclidean space of volume V

$$\int_\Omega \mathrm{d}x = V. \tag{1.2}$$

We further assume that $N(x, y)$ is bounded on Ω

$$|N(x, y)| \leqslant M \tag{1.3}$$

and that it is integrable in Ω with respect to both x and y. The given function $f(x)$ is assumed integrable as well.

It is also useful to introduce the operator $\hat{N}$ and the vectors $|\phi\rangle$ and $|f\rangle$, which correspond to the kernel $N(x, y)$ and functions $\phi(x)$ and $f(x)$. Thus, in obvious notation,

$$N(x, y) = \langle x|\hat{N}|y\rangle \qquad f(x) = \langle x|f\rangle \qquad \text{and} \qquad \phi(x) = \langle x|\phi\rangle. \tag{1.4}$$

In terms of (1.4), we can write Fredholm's equation (1.1) as

$$(\mathbf{1} - \lambda\hat{N})|\phi\rangle = |f\rangle. \tag{1.5}$$

Next, we define the $n \times n$ determinant

$$N\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ & & & \\ y_1 & y_2 & \cdots & y_n \end{pmatrix} = \det_{i,j} N(x_i, y_j) \tag{1.6}$$

where $x_1, \ldots, y_n$ is a set of $2n$ points in Ω. We shall refer to the x_i as the row indices, and to the y_j as the column indices of the symbol on the left-hand side of (1.6). In some of the mathematical literature (1.6) is known as the Fredholm determinant; however, we shall reserve this name, as is customary in vast portions of the physics and mathematics literature, to Fredholm's first series $D(\lambda)$ defined below in (1.9).

Given (1.3), it follows from a theorem due to Hadamard that (1.6) is bounded according to

$$\left| N\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ & & & \\ y_1 & y_2 & \cdots & y_n \end{pmatrix} \right| \leqslant n^{\frac{n}{2}} M^n. \tag{1.7}$$

Fredholm's nth minor is defined by the series

$$D_n\left(\begin{matrix} x_1 & x_2 & \cdots & x_n \\ & & & \\ y_1 & y_2 & \cdots & y_n \end{matrix}\middle|\,\lambda\right) = N\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ & & & \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

$$+ \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int_\Omega N\begin{pmatrix} x_1 & x_2 & \cdots & x_n & s_1 & s_2 & \cdots & s_p \\ & & & & & & & \\ y_1 & y_2 & \cdots & y_n & s_1 & s_2 & \cdots & s_p \end{pmatrix} \mathrm{d}s_1 \ldots \mathrm{d}s_p \tag{1.8}$$

By definition, D_n is completely antisymmetric in the x_i, and also in the y_i. In view of (1.2) and (1.7), it is easy to see that the series (1.8) converges absolutely to an entire function of λ.

Fredholm's first series

$$D(\lambda) = 1 + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int_\Omega N \begin{pmatrix} s_1 & s_2 & \cdots & s_p \\ s_1 & s_2 & \cdots & s_p \end{pmatrix} \mathrm{d}s_1 \ldots \mathrm{d}s_p \tag{1.9}$$

and second series

$$D(x, y; \lambda) = N(x, y) + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int_\Omega N \begin{pmatrix} x & s_1 & s_2 & \cdots & s_p \\ y & s_1 & s_2 & \cdots & s_p \end{pmatrix} \mathrm{d}s_1 \ldots \mathrm{d}s_p \tag{1.10}$$

correspond to setting $n = 0$ and $n = 1$ in (1.8), respectively.

Fredholm's first series (1.9) is, by construction, the functional determinant

$$D(\lambda) = \mathrm{Det}(\mathbf{1} - \lambda \hat{N}) \tag{1.11}$$

of the operator on the left-hand side of (1.5). It is usually known in the literature as the *Fredholm determinant* associated with (1.1), and we shall adhere to this convention here.

From the definitions (1.8) and (1.9) we can prove the important relation

$$\frac{\mathrm{d}^n D(\lambda)}{\mathrm{d}\lambda^n} = (-1)^n \int_\Omega D_n \left(\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} \middle| \lambda \right) \mathrm{d}x_1 \ldots \mathrm{d}x_n \tag{1.12}$$

in a straightforward manner.

The motivation for introducing the minors stems from their important roles in solving Fredholm's equation (1.1) in the most general case. This is briefly reviewed in the appendix. In particular, for values of λ such that $D(\lambda) \neq 0$, the solution of (1.1) is determined by the resolvent kernel $R(x, y; \lambda)$ according to (A.1). $R(x, y; \lambda)$, in turn, is given in (A.2) as $\frac{D(x,y;\lambda)}{D(\lambda)}$. For values of λ such that $D(\lambda) = 0$, the solution is given by (A.6) and (A.10), and involves the higher minors.

The rest of the paper is organized as follows. In the next section we derive our new recursion relations (equations (2.7) and (2.8)) for the minors (1.8). We then solve them and obtain the representation (2.11) for (1.8) as an $n \times n$ determinant over resolvents. In section 3 we provide an interpretation of this representation in terms of non-interacting fermions. Finally, in section 4 we provide an explicit formula (equation (4.9)) for the functional derivative of the nth minor with respect to the kernel.

2. The minor D_n as an $n \times n$ determinant

We start by deriving a couple of integral equations satisfied by D_n [3]. To obtain the first equation, expand each of the determinants in the series (1.8) with respect to the row x_i, and integrate with respect to all the s-variables which occur in that term. It is easy to see, by elementary permutations of rows and columns, that in the pth term, all columns $s_1, \ldots, s_p$ yield the same integrated contribution. Then, after resumming over p, one obtains

$$D_n \left(\begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix} \middle| \lambda \right)$$
$$= \sum_{k=1}^{n} (-1)^{i+k} N(x_i, y_k) D_{n-1} \left(\begin{pmatrix} x_1 & \cdots & \not{x}_i & \cdots & \cdots & x_n \\ y_1 & \cdots & \cdots & \not{y}_k & \cdots & y_n \end{pmatrix} \middle| \lambda \right)$$

$$+\lambda\int_\Omega N(x_i,s)D_n\begin{pmatrix}x_1 & \cdots & (\not{x}_i)s & \cdots & x_n\\ & & & & \\ y_1 & \cdots & \cdots & \cdots & y_n\end{pmatrix}\lambda\Bigg)\,\mathrm{d}s \tag{2.1}$$

where the symbol $\not{x}_i$ in the upper row of D_{n-1} indicates that the row index x_i is to be omitted from the string $x_1,\ldots,x_n$ (and similarly for $\not{y}_k$ in the lower row there), and $(\not{x}_i)s$ indicates that x_i in the upper row of D_n under the integral should be replaced by the integration variable s.

The second integral equation satisfied by D_n is obtained similarly, by expanding each of the determinants in the series (1.8) with respect to the column y_i. One obtains

$$D_n\begin{pmatrix}x_1 & \cdots & x_n\\ & & \\ y_1 & \cdots & y_n\end{pmatrix}\lambda\Bigg)$$

$$=\sum_{k=1}^{n}(-1)^{i+k}N(x_k,y_i)D_{n-1}\begin{pmatrix}x_1 & \cdots & \not{x}_k & \cdots & \cdots & x_n\\ & & & & & \\ y_1 & \cdots & \cdots & \not{y}_i & \cdots & y_n\end{pmatrix}\lambda\Bigg)$$

$$+\lambda\int_\Omega N(s,y_i)D_n\begin{pmatrix}x_1 & \cdots & \cdots & \cdots & x_n\\ & & & & \\ y_1 & \cdots & (\not{y}_i)s & \cdots & y_n\end{pmatrix}\lambda\Bigg)\,\mathrm{d}s. \tag{2.2}$$

We now proceed to derive our own results. We assume henceforth that $D(\lambda)\neq 0$. In this case, according to (A.1), Fredholm's equation (1.1) has a unique solution

$$\phi(x)=f(x)+\lambda\int_\Omega R(x,y;\lambda)f(y)\,\mathrm{d}y. \tag{2.3}$$

From this solution we construct the quantity

$$\Xi_n\begin{pmatrix}x_1 & \cdots & \cdots & \cdots & x_n\\ & & & & \\ y_1 & \cdots & \not{y}_i & \cdots & y_n\end{pmatrix}\lambda\Bigg)=\lambda\int_\Omega D_n\begin{pmatrix}x_1 & \cdots & \cdots & \cdots & x_n\\ & & & & \\ y_1 & \cdots & y_i & \cdots & y_n\end{pmatrix}\lambda\Bigg)\phi(y_i)\,\mathrm{d}y_i. \tag{2.4}$$

Then, we expand the D_n under the integral on the right-hand side of (2.4) according to (2.2). By exploiting the fact that $\lambda\int_\Omega N(x,y)\phi(y)\,\mathrm{d}y=\phi(x)-f(x)=\lambda\int_\Omega R(x,y;\lambda)f(y)\,\mathrm{d}y$ from (1.1) and (2.3), we note that there appears a Ξ_n on the right-hand side of the equation, which cancels the original one on the left, leaving us with the identity

$$\int_\Omega D_n\begin{pmatrix}x_1 & \cdots & \cdots & \cdots & x_n\\ & & & & \\ y_1 & \cdots & y_i & \cdots & y_n\end{pmatrix}\lambda\Bigg)f(y_i)\,\mathrm{d}y_i$$

$$=\sum_{k=1}^{n}(-1)^{i+k}\int_\Omega R(x_k,y_i;\lambda)D_{n-1}\begin{pmatrix}x_1 & \cdots & \not{x}_k & \cdots & \cdots & x_n\\ & & & & & \\ y_1 & \cdots & \cdots & \not{y}_i & \cdots & y_n\end{pmatrix}\lambda\Bigg)f(y_i)\,\mathrm{d}y_i. \tag{2.5}$$

Since this identity holds for all admissible given functions $f(x)$, we conclude that D_n must satisfy the recursion relation

$$D_n\begin{pmatrix}x_1 & \cdots & x_n\\ & & \\ y_1 & \cdots & y_n\end{pmatrix}\lambda\Bigg)=\sum_{k=1}^{n}(-1)^{i+k}R(x_k,y_i;\lambda)D_{n-1}\begin{pmatrix}x_1 & \cdots & \not{x}_k & \cdots & \cdots & x_n\\ & & & & & \\ y_1 & \cdots & \cdots & \not{y}_i & \cdots & y_n\end{pmatrix}\lambda\Bigg) \tag{2.6}$$

or equivalently,

$$D(\lambda)D_n\begin{pmatrix} x_1 & \cdots & x_n \\ & & \\ y_1 & \cdots & y_n \end{pmatrix}\lambda\Bigg) = \sum_{k=1}^{n}(-1)^{i+k}D(x_k, y_i; \lambda)D_{n-1}\begin{pmatrix} x_1 & \cdots & \not{x}_k & \cdots & \cdots & x_n \\ & & & & & \\ y_1 & \cdots & \cdots & \not{y}_i & \cdots & y_n \end{pmatrix}\lambda\Bigg) \tag{2.7}$$

from (A.2). Similarly, from (2.1), and by exploiting the associated (or transposed) Fredholm equation $\psi(x) = g(x) + \lambda\int_\Omega \psi(y)N(y, x)\,\mathrm{d}y$, we obtain the transposed identity

$$D(\lambda)D_n\begin{pmatrix} x_1 & \cdots & x_n \\ & & \\ y_1 & \cdots & y_n \end{pmatrix}\lambda\Bigg) = \sum_{k=1}^{n}(-1)^{i+k}D(x_i, y_k; \lambda)D_{n-1}\begin{pmatrix} x_1 & \cdots & \not{x}_i & \cdots & \cdots & x_n \\ & & & & & \\ y_1 & \cdots & \cdots & \not{y}_k & \cdots & y_n \end{pmatrix}\lambda\Bigg). \tag{2.8}$$

The two identities (2.7) and (2.8) strongly suggest that the quantity

$$\Delta_n\begin{pmatrix} x_1 & \cdots & x_n \\ & & \\ y_1 & \cdots & y_n \end{pmatrix}\lambda\Bigg) = \frac{1}{D(\lambda)}D_n\begin{pmatrix} x_1 & \cdots & x_n \\ & & \\ y_1 & \cdots & y_n \end{pmatrix}\lambda\Bigg) \tag{2.9}$$

is simply the $n \times n$ determinant with entries $R(x_i, y_j; \lambda)$ and corresponding minor

$$\Delta_{n-1}\begin{pmatrix} x_1 & \cdots & \not{x}_k & \cdots & \cdots & x_n \\ & & & & & \\ y_1 & \cdots & \cdots & \not{y}_i & \cdots & y_n \end{pmatrix}\lambda\Bigg).$$

The proof of this proposition by induction is almost trivial: this proposition is indeed the content of (2.7) and (2.8) for $n = 2$:

$$D(\lambda)D_2\begin{pmatrix} x_1 & x_2 \\ & \\ y_1 & y_2 \end{pmatrix}\lambda\Bigg) = D(x_1, y_1; \lambda)D(x_2, y_2; \lambda) - D(x_1, y_2; \lambda)D(x_2, y_1; \lambda). \tag{2.10}$$

(For $n = 1$, (2.7) and (2.8) yield a trivial identity.) Then, by assuming it holds for Δ_{n-1}, we apply it to the Δ_{n-1} which appear on the right-hand sides of (2.7) and (2.8), and thus observe that the latter are just the expansion of an $n \times n$ determinant with entries $R(x_i, y_j; \lambda)$ according to the ith column and ith row, respectively. The proposition of the induction is thus verified for Δ_n as well.

Thus, we have derived our first main result:

$$\frac{1}{D(\lambda)}D_n\begin{pmatrix} x_1 & \cdots & x_n \\ & & \\ y_1 & \cdots & y_n \end{pmatrix}\lambda\Bigg) = \det_{ij} R(x_i, y_j; \lambda). \tag{2.11}$$

Note that for $n = 1$, (2.11) coincides with (A.2), as it should.

3. Interpretation of (2.11) in terms of non-interacting fermions

The determinantal representation (2.11) suggests, due to Wick's theorem for non-interacting fermions, an interpretation of D_n as the correlation function of n fermions and n anti-fermions.

To this end, consider the non-interacting complex Grassmann-valued field $\psi(x)$, living on Ω, with the action

$$S = \int_\Omega \psi^\dagger(x)\,[\delta(x-y) - \lambda N(x,y)]\,\psi(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.1}$$

Its partition function [4] is given by the path integral

$$\mathcal{Z} = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi e^S = \mathrm{Det}(\mathbf{1} - \lambda\hat{N}) = D(\lambda). \tag{3.2}$$

As is well known from the annals of quantum field theory, the non-vanishing correlation functions of (3.2) are those which contain equal numbers of ψ and $\psi^\dagger$, namely,

$$\langle \psi^\dagger(y_1)\psi(x_1)\cdots\psi^\dagger(y_n)\psi(x_n)\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}\psi^\dagger\mathcal{D}\psi e^S[\psi^\dagger(y_1)\psi(x_1)\cdots\psi^\dagger(y_n)\psi(x_n)]. \tag{3.3}$$

The latter are determined according to Wick's theorem as

$$\langle \psi^\dagger(y_1)\psi(x_1)\cdots\psi^\dagger(y_n)\psi(x_n)\rangle = \det_{ij}\langle\psi^\dagger(y_i)\psi(x_j)\rangle \tag{3.4}$$

where the two-point function is

$$\langle\psi^\dagger(y)\psi(x)\rangle = \langle x|\frac{1}{\mathbf{1}-\lambda\hat{N}}|y\rangle. \tag{3.5}$$

Then, note from (3.5) and (A.3) that

$$\int_\Omega N(x,z)\langle\psi^\dagger(y)\psi(z)\rangle\,\mathrm{d}z = \langle x|\frac{\hat{N}}{\mathbf{1}-\lambda\hat{N}}|y\rangle = R(x,y;\lambda). \tag{3.6}$$

Thus, by linearity, we obtain from (3.4) and (3.6) that

$$\langle\psi^\dagger(y_1)(\hat{N}\psi)(x_1)\cdots\psi^\dagger(y_n)(\hat{N}\psi)(x_n)\rangle = \det_{ij} R(x_i,y_j;\lambda) \tag{3.7}$$

where $(\hat{N}\psi)(x) = \int_\Omega N(x,y)\psi(y)\,\mathrm{d}y$. Thus, we interpret the nth minor D_n, according to (2.11) and (3.7), as the multi-fermion correlator

$$\begin{aligned} D_n\begin{pmatrix} x_1 & \cdots & x_n \\ & & \\ y_1 & \cdots & y_n \end{pmatrix}\lambda\Bigg) &= D(\lambda)\langle\psi^\dagger(y_1)(\hat{N}\psi)(x_1)\cdots\psi^\dagger(y_n)(\hat{N}\psi)(x_n)\rangle \\ &= \int \mathcal{D}\psi^\dagger\mathcal{D}\psi e^S[\psi^\dagger(y_1)(\hat{N}\psi)(x_1)\cdots\psi^\dagger(y_n)(\hat{N}\psi)(x_n)] \end{aligned} \tag{3.8}$$

that is, D_n is the $2n$th moment of the Grassmann weight e^S (convoluted against n powers of $\hat{N}$). As such, it might be thought of as some kind of a *continuum* (supplementary) compound matrix associated with $\mathbf{1}-\lambda\hat{N}$ [5]. The latter interpretation might be useful in studying minors of very large order, such that the $2n$ points x_i and y_i become typically dense in Ω.

As a consistency check of (3.8), let us trace it over all coordinates, and see if we recover (1.12). Thus,

$$\begin{aligned} &\int_\Omega D_n\begin{pmatrix} x_1 & \cdots & x_n \\ & & \\ x_1 & \cdots & x_n \end{pmatrix}\lambda\Bigg)\,\mathrm{d}x_1\cdots\mathrm{d}x_n \\ &\quad = D(\lambda)\int_\Omega\langle\psi^\dagger(x_1)(\hat{N}\psi)(x_1)\cdots\psi^\dagger(x_n)(\hat{N}\psi)(x_n)\rangle\,\mathrm{d}x_1\cdots\mathrm{d}x_n \\ &\quad = D(\lambda)\left\langle\left(\int_\Omega\psi^\dagger(x)(\hat{N}\psi)(x)\,\mathrm{d}x\right)^n\right\rangle. \end{aligned} \tag{3.9}$$

From (3.1), (3.2) and (3.3), we see that

$$\left\langle \left(\int_\Omega \psi^\dagger(x)(\hat{N}\psi)(x) \right)^n \right\rangle = \frac{1}{D(\lambda)} \left(-\frac{\mathrm{d}}{\mathrm{d}\lambda} \right)^n D(\lambda). \tag{3.10}$$

Thus,

$$\int_\Omega D_n \begin{pmatrix} x_1 & \cdots & x_n \\ & & \\ x_1 & \cdots & x_n \end{pmatrix} \lambda \, \mathrm{d}x_1 \cdots \mathrm{d}x_n = \left(-\frac{\mathrm{d}}{\mathrm{d}\lambda} \right)^n D(\lambda)$$

in accordance with (1.12).

4. The functional derivative of D_n with respect to the kernel

As we have discussed above, the minors D_n (1.8) determine the solution of the Fredholm equation (1.1). In some applications of (1.1), the kernel $N(x, y)$ may depend on a set of parameters or functions, and it may be important to determine how the solutions vary with these quantities. To this end we have first to determine the functional derivative of the minors D_n with respect to the kernel.

For example, in a recent paper [6], we have calculated the variation of the solution of the Gelfand–Levitan–Marchenko equation with the reflection amplitude of scattering theory, and deduced from it the corresponding variation of the Schrödinger potential and wavefunction.

The minor D_n is expressed in (2.11) in terms of $D(\lambda)$ and $R(x, y; \lambda)$. It is straightforward to obtain the functional derivatives of these two objects with respect to the kernel $N(x, y)$ directly from (1.11) and (A.3). Thus, consider a perturbation $\hat{N} \to \hat{N} + \delta\hat{N}$. From (1.11) we see that under this variation $\delta D(\lambda) = D(\lambda)\delta \log D(\lambda) = -\lambda D(\lambda)\mathrm{tr}\big(\frac{1}{\mathbf{1}-\lambda\hat{N}}\delta\hat{N}\big)$, from which we infer

$$\begin{aligned} \frac{\delta D(\lambda)}{\delta N(a,b)} &= -\lambda D(\lambda)\left[\delta(b-a) + \lambda R(b,a;\lambda)\right] \\ &= -\lambda D(\lambda)\delta(b-a) - \lambda^2 D(b,a;\lambda). \end{aligned} \tag{4.1}$$

Under this variation we also have $\hat{R} \to \hat{R} + \frac{1}{\mathbf{1}-\lambda\hat{N}}\delta\hat{N}\frac{1}{\mathbf{1}-\lambda\hat{N}}$. Consequently

$$\begin{aligned} \frac{\delta R(x,y;\lambda)}{\delta N(a,b)} &= \langle x|(\mathbf{1}+\lambda\hat{R})|a\rangle\langle b|(\mathbf{1}+\lambda\hat{R})|y\rangle \\ &= \left(\delta(x-a) + \lambda R(x,a;\lambda)\right)\left(\delta(b-y) + \lambda R(b,y;\lambda)\right). \end{aligned} \tag{4.2}$$

We can then calculate

$$\frac{\delta}{\delta N(a,b)}\big(D(\lambda)\det_{ij} R(x_i, y_j; \lambda)\big)$$

by applying (4.1) and (4.2) as necessary. The expression we obtain in this way is rather cumbersome, but the plethora of terms thus obtained can be organized into a linear combination of the minors D_n and $D_{n\pm1}$.

Instead of pursuing this line of derivation, we shall now sketch the calculation of the functional derivative of D_n directly from (1.8), by taking the derivative of this series term by term. Thus, consider taking the derivative of the pth term. Let us split the matrix whose determinant

$$\int_\Omega N \begin{pmatrix} x_1 & x_2 & \cdots & x_n & s_{n+1} & s_{n+2} & \cdots & s_{n+p} \\ & & & & & & & \\ y_1 & y_2 & \cdots & y_n & s_{n+1} & s_{n+2} & \cdots & s_{n+p} \end{pmatrix} \mathrm{d}s_{n+1} \cdots \mathrm{d}s_{n+p} \tag{4.3}$$

is being integrated in that term into four blocks, according to NN, NI, IN and II, where I stands for an integrated coordinate index, and N for a non-integrated one. Let us now scan systematically through these blocks.

When the derivative $\frac{\delta}{\delta N(a,b)}$ hits the term $N(x_i, y_j)$ in the NN sector, it produces a factor $\frac{\delta N(x_i,y_j)}{\delta N(a,b)} = \delta(x_i - a)\delta(y_j - b)$ which is multiplied by the minor of $N(x_i, y_j)$ times a sign factor $(-1)^{i+j}$. The total contribution of the NN sector to the derivative is the sum of all these terms:

$$\mathrm{NN} = \sum_{i,j=1}^{n} (-1)^{i+j} \delta(x_i - a)\delta(y_j - b) \times \int_\Omega N \begin{pmatrix} x_1 & \cdots & \not{x}_i & \cdots & \cdots & x_n & s_{n+1} & \cdots & s_{n+p} \\ & & & & & & & & \\ y_1 & \cdots & \cdots & \not{y}_j & \cdots & y_n & s_{n+1} & \cdots & s_{n+p} \end{pmatrix} \mathrm{d}s_{n+1} \cdots \mathrm{d}s_{n+p}. \tag{4.4}$$

Move now to the NI block. When the derivative $\frac{\delta}{\delta N(a,b)}$ hits the term $N(x_i, s_{n+l})$, it produces a factor $\delta(x_i - a)\delta(s_{n+l} - b)$ which is multiplied by the minor of $N(x_i, s_{n+l})$, which contains s_{n+l} as a row index (but not as a column index), times a sign factor $(-1)^{i+n+l}$. Integration over s_{n+l} thus replaces the row index s_{n+l} in that minor by b. Now, permute the row which used to be that of s_{n+l} in that minor, and move it in between the rows corresponding to x_{i-1} and x_{i+1}. This means permuting it across $n + l - i - 1$ rows and costs a sign factor $(-1)^{n+l-i-1}$, which combines with the previous sign factor to (-1), independently of l. Thus, all the p columns which intersect the original row x_i in the NI block make the same contribution to the functional derivative, after integration over the remaining $p - 1$ variables. Finally, summing over all the x_i in the NI block we obtain the total contribution by that block to the functional derivative as

$$\mathrm{NI} = -p \sum_{i=1}^{n} \delta(x_i - a) \times \int_\Omega N \begin{pmatrix} x_1 & \cdots & (\not{x}_i)b & \cdots & x_n & s_{n+1} & \cdots & s_{n+p-1} \\ & & & & & & & \\ y_1 & \cdots & \cdots & \cdots & y_n & s_{n+1} & \cdots & s_{n+p-1} \end{pmatrix} \mathrm{d}s_{n+1} \cdots \mathrm{d}s_{n+p-1}. \tag{4.5}$$

Similarly, the contribution of the entire IN block to the functional derivative is

$$\mathrm{IN} = -p \sum_{j=1}^{n} \delta(y_j - b) \times \int_\Omega N \begin{pmatrix} x_1 & \cdots & \cdots & \cdots & x_n & s_{n+1} & \cdots & s_{n+p-1} \\ & & & & & & & \\ y_1 & \cdots & (\not{y}_j)a & \cdots & y_n & s_{n+1} & \cdots & s_{n+p-1} \end{pmatrix} \mathrm{d}s_{n+1} \cdots \mathrm{d}s_{n+p-1}. \tag{4.6}$$

The last block is II. Clearly, we should discuss the diagonal terms and the non-diagonal terms separately. When $\frac{\delta}{\delta N(a,b)}$ hits the diagonal term $N(s_{n+l}, s_{n+l})$, it produces a factor $\delta(s_{n+l} - a)\delta(s_{n+l} - b)$ times the corresponding diagonal minor, which comes with a positive sign and does not depend on s_{n+l}. Thus, integration over s_{n+l} simply produces a factor $\delta(a - b)$

which multiplies the remaining integral. The latter is the same for all diagonal terms. Thus, the overall contribution of diagonal terms from the II block to the functional derivative is

$$\mathrm{II}_{\mathrm{diag}} = p\delta(a-b)\int_\Omega N\begin{pmatrix} x_1 & \cdots & x_n & s_{n+1} & \cdots & s_{n+p-1} \\ y_1 & \cdots & y_n & s_{n+1} & \cdots & s_{n+p-1}\end{pmatrix} \mathrm{d}s_{n+1}\cdots\mathrm{d}s_{n+p-1}. \tag{4.7}$$

Finally, when $\frac{\delta}{\delta N(a,b)}$ hits the non-diagonal term $N(s_{n+l}, s_{n+k}), l \neq k$, it produces a factor $\delta(s_{n+l}-a)\delta(s_{n+k}-b)$ times the minor of $N(s_{n+l}, s_{n+k})$, which contains s_{n+k} as a row index (but not as a column index), and also contains s_{n+l} as a column index (but not as a row index), times a sign factor $(-1)^{n+k+n+l}$.

Integration over s_{n+l} thus replaces the column index s_{n+l} in that minor by a. Similarly, integration over s_{n+k} replaces the row index s_{n+k} by b. Now, move the row which used to be that of s_{n+k} right below the row x_n, and the column which used to be that of s_{n+l} immediately to the right of the column y_n. These permutations produce a sign factor $(-1)^{k-1+l-1-1}$, which combines with the previous sign simply to (-1). The remaining integral is independent of s_k and s_l and yields the same contribution for all the $p(p-1)$ non-diagonal terms. Thus, their total contribution to the functional derivative is

$$\mathrm{II}_{\text{non-diag}} = -p(p-1)\int_\Omega N\begin{pmatrix} x_1 & \cdots & x_n & b & s_{n+1} & \cdots & s_{n+p-2} \\ y_1 & \cdots & y_n & a & s_{n+1} & \cdots & s_{n+p-2}\end{pmatrix} \mathrm{d}s_{n+1}\cdots\mathrm{d}s_{n+p-2}. \tag{4.8}$$

Gathering all contributions (4.4)–(4.8) together, multiplying their sum by $\frac{(-\lambda)^p}{p!}$ and summing over p, we finally arrive, after some rearrangement of terms, at our second main result:

$$\begin{aligned}
&\frac{\delta}{\delta N(a,b)} D_n\left(\begin{matrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n\end{matrix}\middle|\,\lambda\right) \\
&\quad= \sum_{i,j=1}^{n} (-1)^{i+j}\delta(x_i-a)\delta(y_j-b) D_{n-1}\left(\begin{matrix} x_1 & \cdots & \not{x}_i & \cdots & \cdots & x_n \\ y_1 & \cdots & \cdots & \not{y}_j & \cdots & y_n\end{matrix}\middle|\,\lambda\right) \\
&\qquad+ \lambda\sum_{i=1}^{n}\delta(x_i-a) D_n\left(\begin{matrix} x_1 & \cdots & (\not{x}_i)b & \cdots & x_n \\ y_1 & \cdots & \cdots & \cdots & y_n\end{matrix}\middle|\,\lambda\right) \\
&\qquad+ \lambda\sum_{j=1}^{n}\delta(y_j-b) D_n\left(\begin{matrix} x_1 & \cdots & \ldots & \cdots & x_n \\ y_1 & \cdots & (\not{y}_j)a & \cdots & y_n\end{matrix}\middle|\,\lambda\right) \\
&\qquad- \lambda\delta(a-b) D_n\left(\begin{matrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n\end{matrix}\middle|\,\lambda\right) - \lambda^2 D_{n+1}\left(\begin{matrix} x_1 & \cdots & x_n & b \\ y_1 & \cdots & y_n & a\end{matrix}\middle|\,\lambda\right).
\end{aligned} \tag{4.9}$$

It is gratifying that the functional derivative of D_n is expressed as a relatively simple linear combination of D_n and $D_{n\pm1}$. In particular, we could think of (4.9) as recursively defining D_{n+1} in terms of the lower minors. This is analogous to Jacobi's recursive definition of supplementary compound matrices in the finite-dimensional case [5]. The formula (4.9) for the functional derivative of D_n should coincide, of course, with the expression we would obtain by taking the derivative of $D(\lambda)\det_{ij} R(x_i, y_j; \lambda)$.

As a simple application, let us check (4.9) for $n = 0$ and $n = 1$. For $n = 0$, it yields

$$\frac{\delta D(\lambda)}{\delta N(a,b)} = -\lambda\delta(a-b)D(\lambda) - \lambda^2 D(b,a;\lambda)$$

which coincides with (4.1). Similarly, for $n = 1$, we obtain

$$\frac{\delta D(x,y;\lambda)}{\delta N(a,b)} = \delta(x-a)\delta(y-b)D(\lambda) + \lambda\delta(x-a)D(b,y;\lambda) + \lambda\delta(y-b)D(x,a;\lambda)$$

$$- \lambda\delta(a-b)D(x,y;\lambda) - \lambda^2 D_2\left(\begin{matrix} x & b \\ y & a \end{matrix}\middle|\,\lambda\right). \tag{4.10}$$

Thus, from the last two equations we can derive that

$$\frac{\delta R(x,y;\lambda)}{\delta N(a,b)} = \frac{\delta}{\delta N(a,b)}\left(\frac{D(x,y;\lambda)}{D(\lambda)}\right)$$

$$= \delta(x-a)\delta(y-b) + \lambda\delta(x-a)R(b,y;\lambda) + \lambda\delta(y-b)R(x,a;\lambda)$$

$$+ \lambda^2\left(R(x,y;\lambda)R(b,a;\lambda) - \Delta_2\left(\begin{matrix} x & b \\ y & a \end{matrix}\middle|\,\lambda\right)\right) \tag{4.11}$$

which coincides with (4.2) due to our determinantal representation (2.11).

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Appendix: the solution of Fredholm's equation

The theory of Fredholm's equation (1.1) and its solution is summarized in Fredholm's celebrated three theorems (sometimes referred to collectively as 'Fredholm's alternatives'). The exposition in this appendix will be very telegraphic. We will describe the content of Fredholm's theorems in a semi-quantitative way, sufficient for our purposes, and refer the interested reader to the cited literature for more details.

Broadly speaking, the solution of (1.1) depends on whether $D(\lambda) \neq 0$ or not:

Case (1): $D(\lambda) \neq 0$. In this case, the solution of (1.1) involves the $n = 1$ minor $D(x, y; \lambda)$. For values of λ such that $D(\lambda) \neq 0$, the operator $\mathbf{1} - \lambda\hat{N}$ is invertible, and (1.1) (or, equivalently (1.5)) has a *unique* solution, given by

$$\phi(x) = f(x) + \lambda\int_\Omega R(x,y;\lambda)f(y)\,\mathrm{d}y \tag{A.1}$$

where $R(x, y; \lambda)$ is the resolvent kernel of (1.1), which is given by

$$R(x, y; \lambda) = \frac{D(x, y; \lambda)}{D(\lambda)}. \tag{A.2}$$

From (1.5) we can read off the operator $\hat{R}$, which corresponds to $R(x, y; \lambda) = \langle x|\hat{R}|y\rangle$ as

$$\hat{R} = \frac{1}{\lambda}\left(\frac{1}{\mathbf{1} - \lambda\hat{N}} - \mathbf{1}\right) = \frac{\hat{N}}{1 - \lambda\hat{N}}. \tag{A.3}$$

Thus, from (A.2), (1.12) (at $n = 1$), (1.11) and (A.3) we conclude that

$$\operatorname{tr}\hat{R} = \int_\Omega R(x, x; \lambda)\,\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}\lambda}\log D(\lambda) \tag{A.4}$$

which shows that the poles of $\hat{R}$ as a function of λ are the zeros of $D(\lambda)$.

Case (2): $D(\lambda) = 0$. If $\lambda = \lambda_0$ such that $D(\lambda_0) = 0$, the homogeneous equation

$$\phi(x) = \lambda_0 \int_\Omega N(x, y)\phi(y)\,\mathrm{d}y \tag{A.5}$$

has one or more non-trivial, linearly independent solutions $\Phi_i(x)$, $i = 1, \ldots, \nu$, where $\nu \geqslant 1$. In this case, we say that λ_0 is an eigenvalue of $\hat{N}$ of rank ν, and refer to the functions $\Phi_i(x)$ as the characteristic functions corresponding to the eigenvalue λ_0. (This nomenclature deviates from that of linear algebra, which would refer to $\frac{1}{\lambda_0}$ as the eigenvalue.) Any solution of the homogeneous equation (A.5) is a linear combination of the characteristic functions, i.e. the characteristic functions span $\operatorname{Ker}(1 - \lambda_0\hat{N})$.

As it turns out, the ν characteristic functions are proportional to the νth minor D_ν. More precisely, for a fixed set of 2ν points $x_1, \ldots, y_\nu$, such that

$$D_\nu\begin{pmatrix} x_1 & \cdots & x_\nu \\ & & \\ y_1 & \cdots & y_\nu \end{pmatrix}\lambda_0 \neq 0$$

we have

$$\Phi_i(x) = \frac{D_\nu\begin{pmatrix} x_1 & \cdots & (\not{x}_i)x & \cdots & x_\nu \\ & & & & \\ y_1 & \cdots & y_i & \cdots & y_\nu \end{pmatrix}\lambda_0}{D_\nu\begin{pmatrix} x_1 & \cdots & x_\nu \\ & & \\ y_1 & \cdots & y_\nu \end{pmatrix}\lambda_0} \tag{A.6}$$

where the symbol $(\not{x}_i)x$ indicates that the ith row index x_i in the upper row of the minor in the numerator is to be replaced by the coordinate x. The functions (A.6) are normalized such that

$$\Phi_i(x_k) = \delta_{i,k} \tag{A.7}$$

as can be seen from (1.8), since D_n vanishes when any two of its row (or column) indices coincide.

Since the characteristic functions are expressed in terms of D_ν, it does not vanish identically. In fact, it is the minor of lowest order which does not vanish identically as a function of its 2ν arguments at $\lambda = \lambda_0$, and (A.6) is obtained by setting $n = \nu$ (and $\lambda = \lambda_0$) in (2.1). Furthermore, it follows from (1.12) that λ_0 must be a zero of $D(\lambda)$ of multiplicity, which is greater or equal to ν, since $\frac{\mathrm{d}^\nu D(\lambda)}{\mathrm{d}\lambda^\nu}$ might still vanish at $\lambda = \lambda_0$.

As for solving the inhomogeneous equation (1.1) at $\lambda = \lambda_0$, one proceeds as follows. First, one has to consider the transposed, or associated homogeneous Fredholm equation

$$\psi(x) = \lambda_0 \int_\Omega \psi(y) N(y, x)\, \mathrm{d}y. \tag{A.8}$$

Since its kernel is the transpose of the kernel of (1.1), it has λ_0 as an eigenvalue of the same rank ν. Thus, there are ν independent characteristic solutions $\Psi_i(x)$, which are also expressed in terms of the νth minor D_ν, similar to (A.6), as

$$\Psi_i(x) = \frac{D_\nu \begin{pmatrix} x_1 & \cdots & x_i & \cdots & x_\nu \\ & & & & \\ y_1 & \cdots & (\not{y}_i)x & \cdots & y_\nu \end{pmatrix} \lambda_0}{D_\nu \begin{pmatrix} x_1 & \cdots & x_\nu \\ & & \\ y_1 & \cdots & y_\nu \end{pmatrix} \lambda_0} \tag{A.9}$$

which span $\mathrm{Ker}(1 - \lambda_0 \hat{N}^T)$.

A necessary and sufficient condition for the existence of a solution of (1.1) at $\lambda = \lambda_0$ is then that the given function $f(x)$ be orthogonal to all the characteristic functions $\Psi_i(x)$, i.e. that $\int_\Omega \Psi_i(x) f(x) = 0, i = 1, \ldots, \nu$.

If this condition holds, the solution (which exists) is not unique, since given a particular solution, one can always add to it an arbitrary solution of the homogeneous equation (A.5). The part in the general solution of (1.1) which is linear in $f(x)$ (i.e., a particular solution of (1.1)) is

$$\phi_p(x) = f(x) + \lambda_0 \int_\Omega \frac{D_{\nu+1} \begin{pmatrix} x & x_1 & \cdots & x_\nu \\ & & & \\ y & y_1 & \cdots & y_\nu \end{pmatrix} \lambda_0}{D_\nu \begin{pmatrix} x_1 & \cdots & x_\nu \\ & & \\ y_1 & \cdots & y_\nu \end{pmatrix} \lambda_0} f(y)\, \mathrm{d}y. \tag{A.10}$$

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[4] For the conventions of Grassmannian analysis and path integrals consider any modern standard text book on quantum field theory. In this work, we have followed the conventions of Zinn-Justin J 1996 *Quantum Field Theory and Critical Phenomena* 3rd edn (Oxford: Oxford University Press) chapter 1. See in particular his equation (1.96)

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